

Analysis II

Midterm

CGC/MT/MX/SV

Spring 2024

Questions

For the **multiple choice** questions, we give

- +3 points if your answer is correct,
- 0 points if you give no answer or more than one,
- −1 if your answer is incorrect.

For the **true/false** questions, we give

- +1 points if your answer is correct,
- 0 points if you give no answer or more than one,
- −1 points if your answer is incorrect.

Part I: multiple choice questions

For each question, mark the box corresponding to the correct answer. Each question has **exactly one** correct answer.

Question 1: Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be the function defined by

$$f(x, y) = e^{x-y} - x + e^{2y}.$$

Then

- ☐ the function f admits exactly one local maximum in $\mathbb{R}^2$
- ☐ the function f admits exactly one local minimum in $\mathbb{R}^2$
- ☐ the function f admits a stationary point in $\mathbb{R}^2$ that is not a local extremum
- ☐ the function f does not have stationary points in $\mathbb{R}^2$

Question 2: Let $(\mathbf{a}_n)$ be the sequence of elements in $\mathbb{R}^3$ defined by

$$\mathbf{a}_n = \left(\frac{(-1)^n}{n}, (-1)^n, (-1)^n n \right)^T, \quad \text{for all } n \in \{1, 2, 3, \dots\}.$$

Then

- ☐ the sequence is bounded
- ☐ the sequence has a converging subsequence
- ☐ the sequence has a bounded subsequence
- ☐ the limit $\lim_{n \rightarrow \infty} \mathbf{a}_n$ does not exist

Question 3: Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be the function defined by

$$f(x, y) = \begin{cases} \frac{xy^2}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

Then

- ☐ f is continuous at $(0, 0)$, but $\frac{\partial f}{\partial x}(0, 0)$ does not exist
- ☐ f is not continuous at $(0, 0)$
- ☐ f is continuous at $(0, 0)$, $\frac{\partial f}{\partial x}(0, 0)$ and $\frac{\partial f}{\partial y}(0, 0)$ exist, but f is not differentiable at $(0, 0)$
- ☐ f is differentiable at $(0, 0)$

Question 4: The solution $y(x)$ of the differential equation

$$x y'(x) = \frac{(y(x))^2}{\ln(x) + 1}$$

on the interval $(1, +\infty)$ with initial condition $y(e) = -\frac{1}{2 \ln(2)}$ also satisfies

- | | |
|---|--|
| ☐ $y(e^3) = -\frac{1}{2 \ln(2)}$ | ☐ $y(e^3) = -\frac{1}{\ln(2)}$ |
| ☐ $y(e^3) = -\frac{1}{3 \ln(2)}$ | ☐ $y(e^3) = \frac{1}{4 \ln(2)}$ |

Question 5: Let $f : \mathbb{R}^2 \setminus \{(0, 0)\} \rightarrow \mathbb{R}$ be the function defined by

$$f(x, y) = \frac{y^2 \sin^2(x)}{x^4 + y^2}.$$

Then

☐ $\lim_{(x,y) \rightarrow (0,0)} f(x, y) = 0$

☐ $\lim_{(x,y) \rightarrow (0,0)} f(x, y)$ does not exist

☐ $\lim_{(x,y) \rightarrow (0,0)} f(x, y) = \frac{\pi^2}{4}$

☐ $\lim_{(x,y) \rightarrow (0,0)} f(x, y) = \frac{1}{2}$

Question 6: The solution $y(x)$ of the differential equation

$$y''(x) - 3y'(x) + 2y(x) = 2x^2 - 2x$$

with initial conditions $y(0) = 2$ and $y'(0) = 2$ also satisfies

☐ $y(1) = 5$

☐ $y(1) = 5 + 2e - 2e^2$

☐ $y(1) = 5 - 2e^2$

☐ $y(1) = \frac{5}{4} + 2e$

Question 7: The subset

$$A = \{(x, y) \in \mathbb{R}^2 : x^2 \leq 1 \text{ and } y \leq x^2\} \subset \mathbb{R}^2$$

☐ is bounded and not closed

☐ is closed and bounded

☐ is not bounded and not closed

☐ is closed and not bounded

Part II: true/false questions

For each question, mark the box (without erasing) TRUE if the statement is **always true** and the box FALSE if it is **not always true** (i.e., it is sometimes false).

Question 8: Let A and B be two nonempty subsets of $\mathbb{R}^2$. Then the subset C defined by

$$C = \overline{A} \cup \overline{B}$$

is a closed subset of $\mathbb{R}^2$.

☐ TRUE ☐ FALSE

Question 9: Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a function and let $g : \mathbb{R}^2 \rightarrow \mathbb{R}$ be defined by $g(x, y) = (f(x, y))^2$. If f is not differentiable at $(x, y) = (0, 0)$, then g is not differentiable at $(x, y) = (0, 0)$.

☐ TRUE ☐ FALSE

Question 10: Let D be a nonempty, open, and bounded subset of $\mathbb{R}^n$. If $f : D \rightarrow \mathbb{R}$ is a continuous function, then f attains its global maximum on D .

☐ TRUE ☐ FALSE

Question 11: Let $A = \mathbb{R}^3 \setminus \{(0, 0, 0)\}$. Then the boundary ∂A is the empty set.

☐ TRUE ☐ FALSE

Question 12: Let $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ be a function of class C^3 . Then

$$\frac{\partial^3 f}{\partial x^2 \partial y}(x, y, z) = \frac{\partial^3 f}{\partial x \partial y \partial x}(x, y, z), \quad \text{for all } (x, y, z) \in \mathbb{R}^3.$$

☐ TRUE ☐ FALSE

Question 13: Let $f : \mathbb{R} \rightarrow (0, 1)$ be a function such that $\lim_{t \rightarrow 0} f(t) = L > 0$, then

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 + y^2}{f(x^2 + y^2)} = 0.$$

☐ TRUE ☐ FALSE

Question 14: If $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ is a function of class C^2 , then f admits a tangent plane at each point of its graph.

☐ TRUE ☐ FALSE